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CHARACTERISTICS, CONSERVATION LAWS, AND SYMMETRIES OF THE KINETIC EQUATIONS OF MOTION OF BUBBLES IN A FLUID

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Generalized characteristics and Riemann invariants that are preserved along the characteristics are found for a kinetic model of motion of bubbles in a fluid. Conditions that ensure the hyperbolicity of a set of equations of a bubbly flow are obtained. It is shown that the set of equations of motion has an infinite number of conservation laws. An infinite series of generalized symmetries admitted by the equations is constructed. Solutions that are invariant under the generalized symmetries of solution and describe the propagation of running and simple waves in a bubbly fluid are found.

In modeling the motion of a fluid with gas bubbles, it is important to take into account the effects of collective interaction between bubbles, because they can determine the stability or instability of wave processes in definite regimes of the flow. In some papers [1–4], a kinetic approach to a study of the propagation of concentration waves in bubbly fluids, which is based on the statistical treatment of the interaction between many bubbles, is proposed. Kinetic equations are widely used to describe the motion of and the interaction between charged particles in plasma physics and to describe statistically the flows of a fluid and a gas. In contrast to the motion of charged particles, where the basic long-range forces of interaction are associated with the electric and magnetic fields generated by an ensemble of particles, the interaction between bubbles is associated with the hydrodynamic effects of change in the pressure fields and velocity vector of a fluid in the neighborhood of a given bubble, which is caused by the motion of other bubbles. Russo and Smereka [1] derived kinetic equations for a rarefied bubbly flow, which are similar to Vlasov's equations used in plasma physics. In constructing the model, it was assumed that the bubbles are rigid massless spheres of the same radius. The fluid in which the bubbles move is considered ideal and incompressible, and its flow in the region between the bubbles is assumed to be potential. In addition, the effect of mass forces on the fluid, which is assumed to be quiescent at infinity, is not taken into account. In these conditions, the motion of the bubbles is determined only by the effects of their collective interaction and by the initial conditions. The "rigidity" assumption for the bubbles is approximately fulfilled in actual cases where the bubbles have a rather small radius, and the surface-tension forces preserving the shape of a bubble greatly exceed the hydrodynamic-pressure variations affecting the change in the bubble shape.

It is known that the kinetic energy of the fluid occupying the space between the moving bubbles can be shown in the form of a quadratic dependence of bubble velocities [5]. The coefficients of this quadratic form are calculated through special potentials which describe the flows occurring during motion of one of the bubbles with unit velocity while other bubbles are quiescent. These potentials were approximately calculated in [1] for a rarefied bubbly medium by the asymptotic expansion of the solution of the Laplace equation into a power series of a small parameter, i.e., the ratio of the bubble radius to the average distance between them. This made it possible to obtain a system of ordinary Hamiltonian equations for the coordinates and momenta of the bubbles (the momentum of a bubble is defined as a partial derivative of the kinetic energy of a fluid with respect to the corresponding velocity). Using these equations and the method of deriving Vlasov's equations,

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which was developed in collisionless plasma flow theory, Russo and Smereka derived the following system of integrodifferential equations of a bubbly flow:

$$\frac{\partial f}{\partial t} + \boldsymbol{u}\,\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{F}\,\frac{\partial f}{\partial \boldsymbol{p}} = 0, \qquad \boldsymbol{u} = \frac{2}{\tau\rho_l}\boldsymbol{p} - \frac{3(3\nabla\Phi - \boldsymbol{j})}{\rho_l},$$

$$\boldsymbol{F} = -\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{p}\cdot\boldsymbol{u}), \qquad \Delta\Phi = \operatorname{div}\boldsymbol{j}, \qquad \boldsymbol{j} = \int \boldsymbol{p} f\, d\boldsymbol{p}.$$

Here $f(t, \boldsymbol{x}, \boldsymbol{p})$ is the one-particle distribution function of bubbles on the coordinates and momenta, $\boldsymbol{p}$, $\boldsymbol{u}$, and τ are the momentum, velocity, and volume of the bubble, ρ_l is the density of the fluid, $\Phi(t, \boldsymbol{x})$ is a function which determines the self-consistent field, and the operators ∇ and Δ are calculated with respect to $\boldsymbol{x}$.

The present work studies a model that describes one-dimensional flows. To treat this model, a new theoretical approach, which is based on the generalization of the theory of characteristics and the hyperbolicity concept of a set of equations and developed for a certain class of integrodifferential equations [6, 7], is used. The continuous spectrum of the characteristic velocities of a system of equations of a bubbly flow is shown, and Riemann invariants that are preserved along the characteristics are calculated. Conditions that ensure the hyperbolicity of the system of equations necessary for the flow stability are formulated. Equations that specify the classes of partial solutions, namely, the running and simple waves, are integrated explicitly.

1. Hyperbolicity of the Equations of Motion of Bubbles in a Fluid. In the one-dimensional case, the equation of motion of the bubbles have the form

$$f_t + u f_x - p u_x f_p = 0, \quad j = \int p f\, dp, \quad u = \frac{2p}{\tau\rho_l} - \frac{6j}{\rho_l}. \tag{1.1}$$

Hereafter, the distribution function $f(t, x, p)$ is considered to be either rapidly decreasing at infinity or finite ($f = 0$ for $|p| > B$ and $B > 0$) with respect to the variable p. The integral over the variable p is calculated in the limits $-\infty$ to $+\infty$ (hereinafter, the integration limits in the formulas containing these integrals are omitted).

We introduce the dimensionless variables x', t', p', and f' by the relations

$$x = Lx', \qquad t = (L/U)t', \qquad p = \frac{\tau\rho_l U}{2}p', \qquad f = \frac{2}{3\tau^2\rho_l U}f'$$

where L is proportional to the mean distance between the bubbles, and U is the characteristic velocity. In dimensionless variables, the equations have the form

$$f_t + (p - j)f_x + p j_x f_p = 0, \qquad j = \int p f\, dp \tag{1.2}$$

(the primes are not taken into consideration in the notation of the new variables). We introduce the Lagrangian variable λ, which is preserved along the integral curves of the system of ordinary differential equations:

$$x'(t) = p - j, \qquad p'(t) = p j_x. \tag{1.3}$$

The coordinate surfaces $\lambda(t, x, p)$ are constructed as follows. At the moment $t = 0$, the set of curves $\lambda(x, p) =$ const, where $\lambda_p \neq 0$, is chosen arbitrarily. Then we assume that $\lambda(x, p) =$ const on the surface formed by the integral curves of system (1.3), which pass through the initial curve $\lambda(t, x, p) =$ const. In variables t, x, and λ, the system

$$p_t + (p - j)p_x - p j_x = 0, \quad f_t + (p - j)f_x = 0, \quad j = \int f p p_\lambda\, d\lambda \tag{1.4}$$

is obtained from Eqs. (1.2). If system (1.4) is solved and the functions $f(t, x, \lambda)$ and $p(t, x, \lambda)$ are known, then $p = p(t, x, \lambda)$ is determined from the relation $\lambda = \lambda(t, x, p)$. The substitution of this dependence into f gives the solution of Eqs. (1.2) $f(t, x, p)$. Equations (1.4) can be presented in the general form

$$\boldsymbol{U}_t + A\boldsymbol{U}_x = 0, \qquad \boldsymbol{U} = \boldsymbol{U}(t, x, \lambda), \tag{1.5}$$

where A is the nonlocal operator with respect to the variable λ:

$$A = \begin{pmatrix} (p-j) + p\int f_\lambda p \ldots d\lambda, & -p\int p_\lambda p \ldots d\lambda \\ 0, & (p-j) \end{pmatrix}.$$

Here $T = p\int f_\lambda p \ldots d\lambda$ is the integral operator which acts on an arbitrary function φ according to the rule $T\langle\varphi\rangle = p\int f_\lambda p\varphi\, d\lambda$. The author [6, 7] generalized the concepts of characteristics and hyperbolicity to systems with operator functionals. Let B be the Banach space of the vector functions depending on the variable λ, $\boldsymbol{U}(t,x,\cdot)$, $\boldsymbol{U}_t(t,x,\cdot)$, $\boldsymbol{U}_x(t,x,\cdot) \in B$, A be the linear operator, and $A : B \to B$. Generally, it is assumed that A depends on $\boldsymbol{U}$ (as the operator-valued function on B), x, t, and λ. According to [6, 7], to find the characteristics of system (1.5), one should solve the eigenvalue problem

$$(\boldsymbol{F}^\alpha, A\boldsymbol{\varphi}) = k^\alpha(\boldsymbol{F}, \boldsymbol{\varphi}), \qquad k^\alpha = k^\alpha(x,t) \tag{1.6}$$

for the desired vector functional $\boldsymbol{F}^\alpha \in B'$ and the eigenvalue k^α [$\boldsymbol{\varphi} \in B$ is the trial function, and $(\boldsymbol{F}, \boldsymbol{\varphi})$ is the value of the functional $\boldsymbol{F}$ on $\boldsymbol{\varphi}$]. The characteristic $x = x(t)$ of the family with number α is set by the solution of the differential equation $x'(t) = k^\alpha(x,t)$. The equality

$$(\boldsymbol{F}^\alpha, \boldsymbol{U}_t + k^\alpha \boldsymbol{U}_x) = 0 \tag{1.7}$$

is called a relation on the characteristic [6, 7]. System (1.6) is called a hyperbolic function if all the eigenvalues k^α are real, and the relations on the characteristics (1.7) are equivalent to (1.5). The determination data generalize the corresponding analogs of the classical theory of hyperbolic systems of equations which is concerned with the case where A is the operator in the finite-dimensional space. The basic difference between the infinite-dimensional and the finite-dimensional case is the appearance of the continuous spectra of the characteristic velocities. If the characteristic velocities are limited in absolute value, system (1.5) has the property of finiteness of the perturbation velocity along the x axis.

For system (1.4), the eigenvalue problem (1.6) [$\boldsymbol{\varphi} = (\varphi^1, \varphi^2)^t$ and $\boldsymbol{F} = (F_1, F_2)$] yields two equations for the functionals F_1 and F_2:

$$\begin{gathered} (F_1, (p-j-k)\varphi^1) + \int p f_\lambda \varphi^1\, d\lambda (F_1, p) = 0, \\ (F_2, (p-j-k)\varphi^2) - \int p p_\lambda \varphi^2\, d\lambda (F_1, p) = 0 \end{gathered} \tag{1.8}$$

(the independence of the arbitrary functions φ^1 and φ^2 is used). This allows one to find these functionals

$$(F_1, \varphi) = -\int \frac{p f_\lambda \varphi\, d\lambda}{p-k-j}, \qquad (F_2, \varphi) = \int \frac{p p_\lambda \varphi\, d\lambda}{p-k-j},$$

which correspond to the discrete eigenvalues $k \notin \bar{M}$ and $M = \{p - j,\, p \in \operatorname{supp} f\}$, i.e., the roots of the characteristic equation

$$\chi(k+j) = (1-n) + (k+j)^2 \int \frac{f p_\lambda\, d\lambda}{(p-k-j)^2} = 0. \tag{1.9}$$

Bearing in mind the inequality $0 < n < 1$ (for the model considered, the condition of flow rarefaction means that the quantity n is small), one can conclude that Eq. (1.9) has no real roots belonging to the addition of the set $\bar{M}$ on the real axis. For hyperbolicity of system (1.4), it is necessary that Eq. (1.9) have no complex roots.

To formulate the explicit conditions which guarantee the lack of complex roots, we consider the function $\chi^\pm(z)$ of complex argument z which is analytic in the upper and lower half-planes and apply the principle of argument. If

$$\Delta \arg \chi^+(p) = 0, \tag{1.10}$$

then (1.9) has no complex roots with $\operatorname{Im} k > 0$ [8]. Here $\chi^+(p)$ is the limiting value of the function $\chi^+(z)$ from the upper half-plane on the real axis, and $\Delta \arg$ is the increment of the argument of the complex function on the $p \in (-\infty, \infty)$ axis. If supp$f$ is limited, similarly, we obtain the condition

$$\Delta \arg (\chi^+(p)/\chi^-(p)) = 0 \tag{1.11}$$

where $\chi^-(p)$ is the limiting value from the lower half-plane, and $\Delta \arg$ is the increment of the argument on the carrier function f. If condition (1.10) [or (1.11)] is not satisfied, the Cauchy problem for Eqs. (1.1) is not correct. For the equations linearized on a solution which is independent of x and t, this is shown in [1].

The solutions of problem (1.8) are the continuous family of eigenvalues $k^\lambda = p(t, x, \lambda) - j(t, x)$, where $\lambda \in R$, and the eigenvector functionals $\boldsymbol{F}^{1\lambda}$ and $\boldsymbol{F}^{2\lambda}$ corresponding to k^λ: i.e., $\boldsymbol{F}^{1\lambda} = (0, \delta(\nu - \lambda))$ and $\boldsymbol{F}^{2\lambda} = (F_1^\lambda, F_2^\lambda)$. Here F_i^λ act on an arbitrary smooth function φ according to the rule

$$(F_1^\lambda, \varphi) = (n-1)(\delta(\nu - \lambda), \varphi(\nu)) + p \int \frac{p' f_\nu' (\varphi' - \varphi)\, d\nu}{p' - p}, \quad (F_2^\lambda, \varphi) = -\int \frac{p' p_\nu' \varphi'\, d\nu}{p' - p},$$

and $\delta(\nu - \lambda)$ is the Dirac delta-function. Acting by these functionals on system (1.4), we obtain the characteristic form of the equations

$$f_t + (p - j) f_x = 0, \qquad R_t + (p - j) R_x = 0. \tag{1.12}$$

Here $f(t, x, \lambda)$ and

$$R(t, x, \lambda) = \frac{n-1}{p} + \int \frac{f' p_\nu'\, d\nu}{p' - p} \tag{1.13}$$

are the Riemann invariants which are preserved along the characteristics corresponding to the continuous characteristic spectrum $x = x^\lambda(t)$, $(x^\lambda)'(t) = p(t, x, \lambda) - j(t, x)$, where $\lambda =$ const. In (1.13), the integral is calculated in the sense of the principal value, $p = p(t, x, \lambda)$, $p' = p(t, x, \nu)$, $f' = f(t, x, \nu)$, and $p_\nu' = p_\nu(t, x, \nu)$. We show that conditions (1.10) [or (1.11)]) and the inequality $\chi^\pm(p) \neq 0$ ensure the hyperbolicity of Eqs. (1.4). Here $\chi^\pm(p)$ are the limiting values of the characteristic function on the real axis:

$$\chi^\pm(p) = 1 - n + p^2 \int \frac{f_p'\, dp'}{p' - p} \pm \pi i p^2 f_p. \tag{1.14}$$

Let supp $f = (-\infty, \infty)$. We establish the property of completeness of the family of functionals $\boldsymbol{F}^{1\lambda}$ and $\boldsymbol{F}^{2\lambda}$. To do this, we show that if $(\boldsymbol{F}^{1\lambda}, \varphi) = 0$, $(\boldsymbol{F}^{2\lambda}, \varphi) = 0$, and the function φ satisfies the Hölder condition with respect to the independent variable, then $\varphi = 0$. The equality $\varphi^2 = 0$ from the second relation, with allowance of which the first relation can be written in the form

$$(n-1)\varphi^1 + p \int \frac{p' f_\nu' ((\varphi^1)' - \varphi^1)}{p' - p}\, d\nu = 0.$$

Passing to the integration variable p', we obtain the singular integral equation

$$\left(n - 1 - p \int \frac{p' f_{p'}\, dp'}{p' - p}\right) \varphi^1 + p \int \frac{p' f_{p'} (\varphi^1)'}{p' - p}\, dp' = 0. \tag{1.15}$$

We reduce this equation to the Riemann problem of the theory of analytic functions [9]. In doing so, we introduce the functions $\Psi^+(z)$ and $\Psi^-(z)$, which are analytic in the upper ($\operatorname{Im} z > 0$) and lower ($\operatorname{Im} z < 0$) half-planes, respectively:

$$\Psi^+(z) = \Psi(z), \quad \operatorname{Im} z > 0, \qquad \Psi^-(z) = \Psi(z), \quad \operatorname{Im} z < 0, \qquad \Psi(z) = p \int \frac{p' f_{p'} (\varphi^1)'\, dp}{p' - z}.$$

The limiting values of the functions $\Psi^+(z)$ and $\Psi^-(z)$ on the real axis are calculated by the Sokhotskii–Plemelj formulas:

$$\Psi^\pm(z) = \pm \pi i p^2 f_p \varphi^1 + p \int \frac{p' f_{p'} (\varphi^1)'}{p' - p}\, dp'. \tag{1.16}$$

From Eq. (1.15), with allowance for the equalities (1.14) and (1.16), we obtain the consequence

$$\Psi^{+} = \frac{\chi^{+}}{\chi^{-}}\Psi^{-}, \qquad p \in (-\infty, \infty). \tag{1.17}$$

The Riemann problem of the theory of analytic functions of a complex variable arises: it is required to define the functions $\Psi^{+}(z)$ and $\Psi^{-}(z)$ analytic in the upper and lower complex half-planes, respectively, which satisfy the conjugation condition (1.17) on the real axis and vanish as $|z| \to \infty$. The functions $\chi^{+}(z)$ and $\chi^{-}(z)$ tend to a constant as $|z| \to \infty$ and have no zeros in the upper and lower half-plane according to condition (1.10); therefore, this pair of functions specifies the canonical solution of the Riemann problem [9]. From (1.17), we obtain the jump problem

$$\left(\frac{\Psi}{\chi}\right)^{+} = \left(\frac{\Psi}{\chi}\right)^{-}, \qquad p \in (-\infty, \infty)$$

for the analytic functions $(\Psi/\chi)^{+}(z)$ and $(\Psi/\chi)^{-}(z)$, which has a unique solution in the class of functions vanishing at infinity [8]; therefore $\Psi^{\pm}(z) = 0$. It follows from the equalities (1.15) and (1.16) that $\operatorname{Re}\chi^{+}(p)\varphi^{1}(p) = 0$ and $\operatorname{Im}\chi^{+}(p)\varphi^{1}(p) = 0$. Since $\chi^{\pm}(p) \neq 0$ on the real axis, it is necessary that $\varphi^{1} = 0$. Thus, the completeness of the system of eigenfunctionals and the hyperbolicity of system (1.4) is established if conditions (1.10) and the inequality $\chi^{\pm}(p) \neq 0$ are satisfied.

2. Conservation and Symmetry Laws. System (1.2) admits an infinite number of conservation laws with densities depending in a polynomial manner on the moments of the distribution function. To prove this statement, we consider the function

$$r = \frac{n-1}{\mu} + \int \frac{f'\,dp'}{p'-\mu}. \tag{2.1}$$

We assume that $f \equiv 0$ for $|p| > A > 0$ and μ is a function which depends on x and t and accepts quite large values ($|\mu| > A$) in its domain of definition. From (1.2), we obtain the consequence

$$r_t + (\mu - j)r_x = \left(\frac{1-n}{\mu^2} + \int \frac{f\,dp}{(p-\mu)^2}\right)\left(\mu_t + \left(\frac{\mu^2}{2} - \mu j\right)_x\right). \tag{2.2}$$

Let r be a constant in (2.1), i.e., $r = \xi^{-1}$, where $|\xi| > A$. Then (2.1) defines μ as a function of the variables t, x, and ξ:

$$\mu = -\xi + a_1(t,x) + a_2(t,x)\xi^{-1} + a_3(t,x)\xi^{-2} + \dots .$$

According to (2.2), $\mu(t,x,\xi)$ is the density of the conservation law

$$\mu_t + \left(\frac{\mu^2}{2} - \mu j\right)_x = 0.$$

Expanding $\mu + \xi$ into a power series of ξ^{-1}, we obtain the infinite series of conservation laws, for which the coefficients of the series are the densities of the conservation laws. The first three coefficients have the form

$$a_1 = j = A_1, \quad a_2 = A_2 - A_1^2, \qquad a_3 = A_3 - 3A_2A_1 + 2A_1^3 \quad \left(A_i = \int p^i f\,dp\right).$$

Differentiating (2.1), we sequentially determine the following coefficients depending on higher moments. To prove that system (1.2) has an infinite series of conservation laws, the boundedness of the carrier function f was used. However, if one approximates an arbitrary, rapidly decreasing at infinity distribution function by a sequence of functions with limited carriers, realizing a limiting transition in each divergent equation, one obtains an infinite series of conservation laws for an arbitrary f.

We note that system (1.4) is reduced in the Hamiltonian form

$$p_t + \left(\frac{\delta h}{\delta H(\lambda)}\right)_x = 0, \qquad H_t + \left(\frac{\delta h}{\delta p(\lambda)}\right)_x = 0, \tag{2.3}$$

if the new desired function $H(t,x,\lambda) = f(t,x,\lambda)p_\lambda(t,x,\lambda)$ and the Hamiltonian with the density

$$h = \frac{1}{2}\left(\int Hp^2\,d\lambda - j^2\right) \tag{2.4}$$

are introduced. The variational derivatives $\delta h/\delta H(\lambda)$ and $\delta h/\delta p(\lambda)$ of the Hamiltonian density (2.4) are calculated in the form

$$\frac{\delta h}{\delta H(\lambda)} = \frac{1}{2}p^2 - pj, \qquad \frac{\delta h}{\delta p(\lambda)} = H(p-j).$$

It follows from the existence of the Hamiltonian and the infinite series of conservation laws that system (2.3) admits an infinite number of first-order generalized symmetries (the one-parameter Lie–Bäcklund groups) [9–11]. If a_i is the density of the conservation law, the equations

$$p_\tau = \left(\frac{\delta a_i}{\delta H(\lambda)}\right)_x, \qquad H_\tau = \left(\frac{\delta a_i}{\delta p(\lambda)}\right)_x \tag{2.5}$$

determine the symmetry for (2.3). This means, in particular, that the one-parameter family of solutions of the Cauchy problem for system (2.5) $p(\tau,t,x,\lambda)$ and $H(\tau,t,x,\lambda)$ (t is the parameter) which satisfies the initial conditions $p(0,t,x,\lambda) = p(t,x,\lambda)$ and $H(0,t,x,\lambda) = H(t,x,\lambda)$ is simultaneously the one-parameter family of solutions of system (2.3) (τ is the parameter), if the initial functions $p(t,x,\lambda)$ and $H(t,x,\lambda)$ satisfy (2.3).

From (2.5), we obtain, in particular, the equations specifying the symmetry which corresponds to the density a_3 of the conservation law:

$$p_\tau = (p^3 - 3p^2j - 3pA_2 + 6pj^2)_x, \quad H_\tau = (H(3p^2 - 6pj - 3A_2 + 6j^2))_x \quad \left(A_i = \int p^iH\,d\lambda\right). \tag{2.6}$$

It is easy to see that system (1.2) also admits the point transformations of transfer relative to t and x: $t \to t+a$ and $x \to x+b$, and the two-parameter extension of the variables x, t, p, and f: $t \to at$, $x \to bx$, $p \to ba^{-1}p$, and $f \to ab^{-1}f$. Equations (1.2) are invariant under the Galilei transformation (the transition to the coordinate system moving with a constant velocity). This circumstance is explained by the fact that the assumption of the zero velocity of the fluid at infinity underlies the derivation of the system of equations, which causes the invariance of the equations of motion relative to the Galilei transformation.

3. Running Waves. A solution of the form $f = f(\zeta,p)$, where $\zeta = x - Dt$, describes a running wave propagating with a constant velocity D. It is convenient to transform the equation of running waves

$$(p - j - D)f_\zeta + pj_\zeta f_p = 0 \tag{3.1}$$

by using j as the independent variable (assuming that $j_\zeta \neq 0$):

$$(p - j - D)f_j + pf_p = 0. \tag{3.2}$$

Equation (3.2) is integrated as follows:

$$f = \Phi(\eta), \qquad 2\eta = p^2 - 2(j+D)p. \tag{3.3}$$

The above solution takes on constant values on the characteristics $\eta = \text{const}$ (Fig. 1 shows the pattern of the characteristics of Eq. (3.2) in the plane p, j). We consider the Cauchy problem

$$f(\zeta_0,p) = f_0(p), \qquad j_0 = \int pf_0\,dp. \tag{3.4}$$

Condition (3.4) ensures a continuous contiguity of the running wave to the known stationary background [$f_0(p)$ is a specified distribution function] over which the wave propagates. One can see in Fig. 1 that, if $j_0 + D > 0$, the solution of the Cauchy problem is determined for $j \geqslant j_0$ only in the domains Ω_1 and Ω_2 [the addition to the domain Ω_3 in the half-plane $j \geqslant j_0$, and the curve DAE given by the equation $\eta = 2^{-1}(j_0+D)^2$ is the boundary of Ω_3]. In the domain Ω_3, the solution should be found by using additional equations. In the case where $j_0 + D > 0$, the Cauchy problem (3.4) is incorrect in the direction of the decreasing variable j ($j \leqslant j_0$). Indeed, each characteristic $\eta = \eta_1 < 0$ intersects the initial straight line $j = j_0$ at two points B and C; therefore, the initial function $f_0(p)$ cannot be arbitrary and should take on the same values at these points.

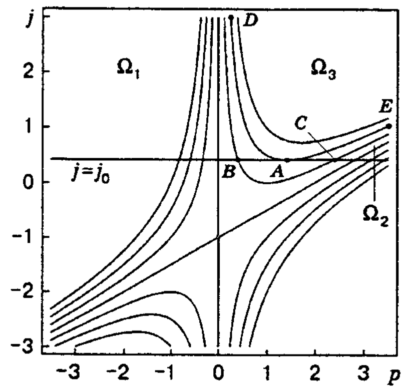

Fig. 1

Similarly, for $j_0 + D < 0$, one can solve the Cauchy problem (3.4) only in the direction of the decreasing j ($j \leqslant j_0$).

Using (3.3), for $\zeta = \zeta_0$, we shall define $\Phi(\eta)$ for $p \leqslant j_0 + D$ and $p \geqslant j_0 + D$ (the corresponding functions are denoted by Φ^- and Φ^+):

$$\Phi^+(\eta) = f_0(j_0 + D + \sqrt{2\eta + (j_0 + D)^2}\,), \quad \Phi^-(\eta) = f_0(j_0 + D - \sqrt{2\eta + (j_0 + D)^2}\,). \tag{3.5}$$

According to (3.3), for Ω_1 and Ω_2, the running wave is defined by the relations

$$\begin{gathered} f = \Phi^+(2^{-1}p^2 - (j + D)p), \quad p \geqslant j + D + \sqrt{(j + D)^2 - (j_0 + D)^2}, \\ f = \Phi^-(2^{-1}p^2 - (j + D)p), \quad p \leqslant j + D - \sqrt{(j + D)^2 - (j_0 + D)^2}. \end{gathered} \tag{3.6}$$

To find the distribution function in the domain Ω_3, we transform the equality

$$j = \int pf\, dp \tag{3.7}$$

into an integral equation for the desired function $\Phi(\eta)$, where $\eta \in (-(j_1 + D)^2/2, -(j_0 + D)^2/2)$. Here it is assumed that $j_0 \leqslant j \leqslant j_1$ in the domain $\zeta_0 \leqslant \zeta \leqslant \zeta_1$ occupied by the running wave. After the introduction of the integration variables η into (3.7), instead of p, we obtain the Abel integral equation

$$\begin{gathered} \int_s^{s_0} \frac{\Phi(\eta)\, d\eta}{\sqrt{\eta - s}} = F(s), \\ F(s) = (2\sqrt{-s})^{-1}\Big[\sqrt{-2s} - D - \int_{s_0}^{\infty} (\Phi^+(\eta) - \Phi^-(\eta))\, d\eta - 2\sqrt{-s}\int_{s_0}^{\infty} \frac{(\Phi^+(\eta) + \Phi^-(\eta))\, d\eta}{\sqrt{\eta - s}}\Big], \end{gathered} \tag{3.8}$$

where $s = -2^{-1}(j + D)^2$ and $s_0 = -2^{-1}(j_0 + D)^2$. The solution of the Abel equation has the form

$$\Phi(\eta) = \frac{1}{\pi}\Big[\frac{F(s_0)}{\sqrt{s_0 - \eta}} - \int_\eta^{s_0} \frac{F'(s)\, ds}{\sqrt{s - \eta}}\Big]. \tag{3.9}$$

We note that the second relation in (3.4) is equivalent to $F(s_0) = 0$; as a consequence, the first term in formula (3.9) is equal to zero. The distribution function is defined by formulas (3.8) and (3.9) in the domain Ω_3, where $-\sqrt{(j + D)^2 - (j_0 + D)^2} \leqslant p - j - D \leqslant \sqrt{(j + D)^2 - (j_0 + D)^2}$. The resulting class of solutions depends on an arbitrary function $j(\zeta)$. The running wave which propagates with velocity $D < -j_0$ can be constructed similarly.

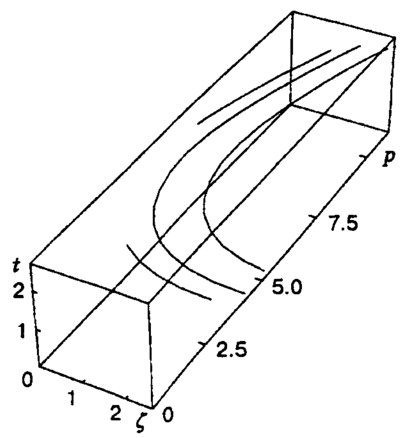

Fig. 2

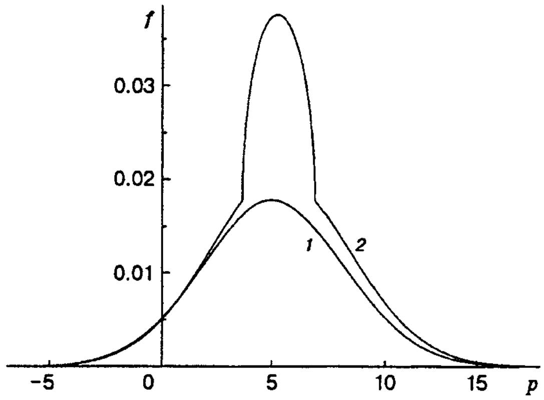

Fig. 3

Integrating (3.2) over the variable p, we obtain

$$1 - n = (j + D)n_j. \tag{3.10}$$

It follows from (3.10) that the density of the bubble n increases with $|j|$ for two types of running wave. The constructed solution of the running-wave type describes the penetration of a portion of bubbles into the unperturbed area. We show it by reference to the pattern of the trajectories.

In a coordinate system moving together with the wave, the trajectories of the bubbles are determined by the system of ordinary differential equations

$$\frac{d\zeta}{dt} = p - j - D, \qquad \frac{dp}{dt} = j_\zeta p.$$

It is easy to see that $\eta = \text{const}$ is the integral of the system. This makes it possible to use Fig. 1 to analyze the motion pattern. In the domains Ω_1 and Ω_2, the quantity $p - j - D$ does not vanish, and the trajectories intersect the running-wave region in opposite directions ($p - j - D$ has different signs in Ω_1 and Ω_2). For particles with a negative parameter $\eta < s_0$ (the domain Ω_3), $p - j - D$ changes sign during particle motion along the trajectory. These bubbles enter the running-wave region through the front $\zeta = \zeta_1$, and their relative velocity changes sign at some point of the trajectory (here, the coordinate ζ reaches a value extreme for the given trajectory), and then the bubble returns to the front $\zeta = \zeta_1$ and leaves the running-wave zone. The emergence of these bubbles in the running-wave region increases the density n. Figure 2 shows the trajectories of the bubbles in the domain of definition of a running wave in the space t, ζ, p.

Choosing an arbitrary function $j(\zeta)$, one can construct periodic running waves, solitons, etc. It suffices to choose j as a periodic function ζ in the first case or a function satisfying the relation $j(\zeta_0) = j(\zeta_1)$ in the second. The soliton solution describes the motion of the bubbles trapped in the running wave. In this case, the trajectories of the bubbles which correspond to the negative parameter $\eta < s_0$ remain in the running-wave region $\zeta_0 \leqslant x - Dt \leqslant \zeta_1$ for all values of t.

Below, we present the explicit formulas for a running wave propagating over the stationary background given by the Maxwellian distribution:

$$f_0(p) = \frac{n_0}{\sqrt{2\pi T}} \exp\left(-(2T)^{-1}(p - a)^2\right).$$

Here n_0, T, and a are the specified constants. It is easy to verify that, for this stationary distribution, we have $n = n_0$ and $j = an_0$. We consider a running wave propagating with the velocity $D = a - j_0$. According to (3.3) and (3.5), for $\zeta = \zeta_0$, we have

$$2\eta = p^2 - 2ap = (p - a)^2 - a^2,$$

$$\Phi^+(\eta) = \Phi^-(\eta) = \frac{n_0}{\sqrt{2\pi T}} \exp\left(-(2T)^{-1}(2\eta + a^2)\right), \qquad \eta \geqslant -\frac{a^2}{2}.$$

In the domains $p \geqslant j + D + \sqrt{(j+D)^2 - a^2}$ and $p \leqslant j + D - \sqrt{(j+D)^2 - a^2}$, according to (3.6), the distribution function has the form

$$f = \frac{n_0}{\sqrt{2\pi T}} \exp\left(-(2T)^{-1}(p^2 - 2(j+D)p + a^2)\right). \tag{3.11}$$

Using formulas (3.5)-(3.9), we find the distribution function in the domain $j + D - \sqrt{(j+D)^2 - a^2} \leqslant p \leqslant j + D + \sqrt{(j+D)^2 - a^2}$:

$$f = \Phi(p^2/2 - (j+D)p),$$
$$\Phi(\eta) = \frac{n_0}{\sqrt{2\pi T}}\left(1 - \operatorname{erf}\left(\left(-\frac{a^2 + 2\eta}{2T}\right)^{1/2}\right)\right) \exp\left(-\frac{a^2 + 2\eta}{2T}\right) - (2\eta)^{-1}(1 - n_0)\sqrt{-(a^2 + 2\eta)},$$
$$\eta \leqslant -a^2/2. \tag{3.12}$$

The running wave is defined completely by the equalities (3.11) and (3.12). Here $j = j(\zeta)$ remains an arbitrary function taking values on the semi-axis (an_0, ∞) (the case $j \geqslant j_0$). In Fig. 3, the distribution function is plotted in the forward front (curve 1) and at the cross section $\zeta = \text{const}$, $\zeta \in (\zeta_0, \zeta_1)$ (curve 2) in the domain of definition of the running wave. One can see that the penetration of a new portion of bubbles into the unperturbed region entails a considerable increase in the distribution function on the interval corresponding to the domain Ω_3.

4. Solutions Invariant under Generalized Symmetries. As shown in Sec. 2, the system of bubble-flow equations admits generalized symmetries determined by integrodifferential equations. Generalizing the methods developed in the classical group analysis of differential equations [9], one can construct solutions invariant under integrodifferential symmetries. Here we construct a solution invariant under the symmetry (2.6).

We write the determining equations for the symmetries of Eq. (1.2) without using system (2.5). Let the symmetry be set by the equation

$$f_\tau = F(f, f_x, p, x, t). \tag{4.1}$$

Here $F(f, p, x, t)$ is the nonlinear functional above $f_x(\cdot, x, t)$, $f(\cdot, x, t) \in B$ which depends on p, x, and t. We introduce the functional $g(f, p) = (1/2)p^2 - pj$. Equations (1.2) are written in the form

$$f_t + g_p f_x - D_x g f_p = 0, \tag{4.2}$$

where D_x is the operator of the full derivative with respect to x:

$$D_x g = \frac{\partial g}{\partial x} + \frac{\delta g}{\delta f}\langle f_x\rangle + \frac{\delta g}{\delta f_x}\langle f_{xx}\rangle + \dots\ . \tag{4.3}$$

In formula (4.3), $\delta g/\delta f$ and $\delta g/\delta f_x$ are the Frechet derivatives with respect to the arguments f and f_x. The determining equations for F are derived after differentiation of (4.2) with respect to τ:

$$D_t F + g_p D_x F - D_x(g) F_p + g_{\tau p} f_x - D_x(g_\tau) f_p = 0. \tag{4.4}$$

Here D_t is introduced as was done for (4.3) with replacement of x by t, and

$$g_\tau = \frac{\delta g}{\delta f}\langle f_\tau\rangle = \frac{\delta g}{\delta f}\langle F\rangle.$$

Equation (4.4) should be satisfied on the solutions of system (4.2). If one searches for F in a special form

$$F = G_p f_x - D_x(G) f_p, \qquad G = G(f, p, x, t), \tag{4.5}$$

one obtains, from (4.4), the determining equation for the functional G

$$D_tG + g_pD_xG - D_x(g)G_p = g_\tau. \tag{4.6}$$

As (4.4), Eq. (4.6) holds by virtue of (4.2) and its consequences. The symmetries of Eq. (4.2) correspond to the symmetries (2.5) of systems (2.3). Using the transformations relating these equations, we find the class of solutions of Eq. (4.6):

$$G_i = \frac{\delta a_i}{\delta f} = \sum_{j=1}^{i} \frac{\partial a_i}{\partial A_j} p^j.$$

In particular, for $i = 3$, the functional

$$G = p^3 - 3A_1p^2 - 3A_2p + 6A_1^2p \tag{4.7}$$

is the solution of (4.6). By definition, the invariant solutions of Eq. (4.2) satisfy the steady-state equations (4.1):

$$F = G_pf_x - D_x(G)f_p = 0.$$

As a result, we have the functional dependence $f = \Phi(G,t)$. Using the relation $g_\tau = 0$, which is valid for invariant solutions, from (4.6) and (4.7) we obtain the system of equations for the moments A_1 and A_2:

$$A_{1t} + A_{2x} - 3A_1A_{1x} = 0, \quad A_{2t} + 3A_1A_{2x} + (A_2 - 10A_1^2)A_{1x} = 0. \tag{4.8}$$

Equations (4.2) will be satisfied if $\Phi_t = 0$ and $f = \Phi(G)$ satisfies the equality

$$j = A_1 = \int_{-\infty}^{\infty} p\Phi(p^3 - 3A_1p^2 - 3A_2p + 6A_1^2p)\,dp. \tag{4.9}$$

Since (4.9) sets the relation between A_1 and A_2 if the function Φ is known, the solution of Eqs. (4.8) should be searched for in the class of simple waves. We introduce $B = A_2A_1^{-2}$. System (4.8) is hyperbolic in the domain $B > 1$ and is reduced to the Riemann invariants:

$$r_t + A_1\sqrt{B-1}\,r_x = 0, \qquad l_t - A_1\sqrt{B-1}\,l_x = 0,$$
$$r = A_1^2\left|\sqrt{B-1} - 2^{-1}\right|^{2/3}(\sqrt{B-1}+1)^{4/3}, \quad l = A_1^2\left|\sqrt{B-1} - 1\right|^{4/3}(\sqrt{B-1}+2^{-1})^{2/3}. \tag{4.10}$$

It follows from (4.10) that the simple wave satisfies either the relation $r = r_0 = \text{const}$ or $l = l_0 = \text{const}$. In the first case, $A_1 = \sqrt{r_0}|\sqrt{B-1} - 2^{-1}|^{-1/3}(\sqrt{B-1}+1)^{-2/3}$, and the function B is the solution of the equation

$$B_t - \sqrt{r_0}\left|\sqrt{B-1} - 2^{-1}\right|^{-1/3}(\sqrt{B-1}+1)^{-2/3}\sqrt{B-1}\,B_x = 0. \tag{4.11}$$

In the second case, $A_1 = \sqrt{l_0}\left|\sqrt{B-1} - 1\right|^{-2/3}\left|\sqrt{B-1} + 2^{-1}\right|^{-1/3}$, and the function B satisfies the equation

$$B_t + \sqrt{l_0}\left|\sqrt{B-1} - 1\right|^{-2/3}\left|\sqrt{B-1} + 2^{-1}\right|^{-1/3}\sqrt{B-1}\,B_x = 0. \tag{4.12}$$

In what follows, we assume that $A_1 = A_1(B)$, where $A_1(B)$ is one of the two above functions. We transform (4.9) into an integral equation relative to the function Φ. In the integral (4.9), it is convenient to pass to the integration variable G. Since

$$dG = (3p^2 - 6A_1p + 3(2-B)A_1^2)\,dp, \tag{4.13}$$

the correspondence of the new and old variables is one-to-one on the intervals divided by the roots of the square polynomial in parentheses. It follows from (4.13) that $dG/dp > 0$ for $p > A_1(1+\sqrt{B-1})$ and $p < A_1(1-\sqrt{B-1})$. For $p \in (A_1(1-\sqrt{B-1})$, where $A_1(1+\sqrt{B-1}))$, the derivative dG/dp is negative. The function G takes on the value $G_1(B) = A_1^3(4 - 3B - 2(B-1)^{3/2})$ at the point of local minimum

$[p = A_1(1+\sqrt{B-1})]$ and the value $G_2(B) = A_1^3(4-3B+2(B-1)^{3/2})$ at the point of local maximum (for definiteness, we assume that $A_1 > 0$).

After the variable has been replaced, relation (4.9) takes the form

$$A_1(B) = \int_{-\infty}^{G_2(B)} \frac{\Phi(G)p_1}{G'(p_1)}\,dG + \int_{G_1(B)}^{\infty} \frac{\Phi(G)p_3}{G'(p_3)}\,dG - \int_{G_1(B)}^{G_2(B)} \frac{\Phi(G)p_2}{G'(p_2)}\,dG. \tag{4.14}$$

Here $p = p_1(G,B)$ is the branch of the inverse function, i.e., the root of Eq. (4.7) on the monotonicity site $G \in (-\infty, G_2)$ and $p = p_3(G,B)$ is a similar branch determined on the monotonicity site of a $G \in (G_1, \infty)$, and the branch $p = p_2(G,B)$ is determined on the site $G \in (G_1, G_2)$. On the last interval, the inequality $p_1 < p_2 < p_3$ is satisfied. Using the functions p_i, one can present (4.7) in the form

$$\chi(p) = 0, \quad \chi(p) = (p-p_1(G,B))(p-p_2(G,B))(p-p_3(G,B)).$$

Since $\chi'(p_i) = G'(p_i) = \prod_{k\neq i}(p_i - p_k)$, it is easy to relate the kernels of the integral equation (4.14):

$$-\frac{p_2}{G'(p_2)} = \frac{p_1}{G'(p_1)} + \frac{p_3}{G'(p_3)}.$$

With allowance for this relation, (4.14) takes the form

$$A_1(B) = \int_{-\infty}^{G_1(B)} \frac{\Phi(G)p_1(G,B)}{\chi'(p_1(G,B))}\,dG + \int_{G_2(B)}^{\infty} \frac{\Phi(G)p_3(G,B)}{\chi'(p_3(G,B))}\,dG$$

$$+2\int_{G_1(B)}^{G_2(B)} \Phi(G)\Big(\frac{p_1(G,B)}{\chi'(p_1)(G,B)} + \frac{p_3(G,B)}{\chi'(p_3)(G,B)}\Big)\,dG. \tag{4.15}$$

We show that, using Eqs. (4.15), one can find the distribution function in a simple wave of small amplitude. Let B vary on the interval (B_0, B_1) in the simple-wave region; here the function $G_2(B)$ monotonically varies from G_{20} to G_{21}, and the function $G_1(B)$ takes on the values on the interval (G_{10}, G_{11}), and $G_{11} < G_{20}$. We define $\Phi(G)$ arbitrarily outside the interval (G_{20}, G_{21}) and find the solution of the integral equation

$$2\int_{G_{20}}^{G_2(B)} \Big(\frac{p_1(G,B)}{\chi'(p_1(G,B))} + \frac{p_3(G,B)}{\chi'(p_3(G,B))}\Big)\Phi(G)\,dG + \int_{G_2(B)}^{G_{21}} \frac{p_3(G,B)}{\chi'(p_3(G,B))}\Phi(G)\,dG$$

$$= A_1(B) - \int_{-\infty}^{G_1(B)} \frac{p_1(G,B)}{\chi'(p_1(G,B))}\Phi(G)\,dG - \int_{G_{21}}^{\infty} \frac{p_3(G,B)}{\chi'(p_3(G,B))}\Phi(G)\,dG$$

$$-2\int_{G_1(B)}^{G_{20}} \Big(\frac{p_1(G,B)}{\chi'(p_1)(G,B)} + \frac{p_3(G,B)}{\chi'(p_3)(G,B)}\Big)\Phi(G)\,dG$$

inside the interval. This is the linear equation of the first kind with an integral operator of the form

$$\int_{G_{20}}^{S} (K_1(G,S) + K_2(G,S))\Phi(G)\,dG + \int_{S}^{G_{21}} K_3(G,S)\Phi(G)\,dG = \Psi(S). \tag{4.16}$$

Here $S = G_2(B)$ [accordingly, $B = G_2^{-1}(S)$], $\Psi(S)$ is the known function, and $K_i(G,S)$ are the known kernels of the integral operators.

We show that (4.16) is reduced to an integral equation of the second kind resolved by the iteration method. The kernel of the integral operator $K_2(G,S)$ has no singularities in the integration domain, because $\chi'(p_3) \neq 0$ $[G_1(B) < G_{11}]$. However, the function $K_1(G,S)$ reverts to infinity for $G = S$ by virtue of the fact that $\chi'(p_1) = 0$ for $G = G_2(B)$. We separate the principal term of the singularity of the kernel $K_1(G,S)$. Relation (4.7) can be written in the form

$$(p - p_*)^2 = \frac{G_2(B) - G}{A_1\sqrt{B-1}}\left(1 + \frac{p - p_*}{3A_1\sqrt{B-1} - (p - p_*)}\right). \tag{4.17}$$

Here $p_* = A_1(1 - \sqrt{B-1})$. One can expand $p_i - p_*$ into a power series of $(G_2(B) - G)^{1/2}$ by iterating (4.17). Using the first terms of the expansion, we obtain the representation

$$\frac{1}{p_2(G,B) - p_1(G,B)} = \frac{2\sqrt{A_1}(B-1)^{1/4}}{\sqrt{G_2(B) - G}} + K_3(G,S), \tag{4.18}$$

where $K_3(G,S)$ is a continuously differentiable function having first-order zeros for $G = S$. The second cofactor in the representation of the kernel $K_1(G,S)$

$$\frac{p_3}{p_3 - p_1} = \frac{1 + 2\sqrt{B-1}}{3\sqrt{B-1}} + K_4(G,S) \tag{4.19}$$

has no singularities. The function $K_4(G,S)$ has the same properties as the function $K_3(G,S)$. Multiplied expressions (4.18) and (4.19), we obtain

$$K_1(G,S) = \frac{\eta(S)}{\sqrt{S-G}} + \sqrt{S-G}K_5(G,S) + K_6(G,S). \tag{4.20}$$

Here $K_5(G,S)$ and $K_6(G,S)$ are continuously differentiable functions, $K_6(G,S) = 0$ for $G = S$, and

$$\eta(S) = \frac{2\sqrt{A_1}(1 + 2\sqrt{B-1})}{3(B-1)^{1/4}}.$$

The substitution of (4.20) into (4.16) gives the integral equation

$$\int_{G_{20}}^{S} \frac{\Phi(G)}{\sqrt{S-G}}\,dG = \Psi_1(S) = \int_{G_{20}}^{S}(\sqrt{S-G}L_1(G,S) + L_2(G,S))\Phi(G)\,dG + \int_{S}^{G_{21}} L_3(G,S)\Phi(G)\,dG + (2\eta(S))^{-1}\Psi(S)$$

with the continuously differentiable kernels $L_i(G,S)$.

Inverting the Abel integral operator, we obtain an equation of the second kind

$$\Phi(G) = \frac{1}{\pi}\int_{G_{20}}^{G} \frac{\Psi_1'(S)}{\sqrt{G-S}}\,dS, \tag{4.21}$$

which is unambiguously resolved by the iteration method for small values of $|G_{21} - G_{20}|$.

If the distribution function $\Phi(G)$ is found from (4.21), the solution which is invariant under the generalized symmetry is completely determined by any solution $B = B(x,t)$ of Eqs. (4.11) [or (4.12)].

In the resulting simple wave, the particle trajectories are the integral curves of the system

$$\frac{dx}{dt} = p - j, \qquad \frac{dp}{dt} = pj_x. \tag{4.22}$$

In the simple-wave region, it is convenient to search for the trajectories in an implicit form: $k(x,t) = k_0(t)$ and $p = p_0(t)$, where $k = k(x,t)$ is the angular declination of the characteristics of the simple wave. For $k_x \neq 0$, resolving the equation $k(x,t) = k_0(t)$ relative to x, one can find the coordinate x of the particle $x = x_0(t)$. Since the function $k(x,t)$ satisfies the equation $k_t + kk_x = 0$ in a simple wave, the equalities

$$\frac{dk}{dt} = (p - j - k)k_x, \qquad \frac{dp}{dt} = pj_k k_x \tag{4.23}$$

follow from (4.22). System (4.23) has the integral $G(p,k) = \text{const}$ owing to the fact that Eq. (4.6) takes the following form in the simple wave:

$$(p - j - k)G_k + pj_kG_p = 0. \tag{4.24}$$

It follows from (4.24) that the extrema of the function $G(p)$ correspond to the points of rotation of the trajectories in a coordinate system moving together with the wave (the points where $p - j - k$ changes sign). Hence, the constructed solution $f = f(p,k)$ describes the flow with critical layers: the velocity of the simple wave k coincides with the particle velocity $p - j$ at definite points of the trajectories.

We note that the running wave constructed in Sec. 3 may be considered as a solution invariant under the generalized symmetry. For this solution, the symmetry is set by Eqs. (4.1) and (4.5) with the functional $G = p^2/2 - (j + D)p$.

Conclusion. The system of kinetic equations for the one-dimensional motion of bubbles in a fluid that was derived by G. Russo and P. Smereka has several interesting properties. These equations are reduced to an integrodifferential system which is hyperbolic (in the meaning of [6, 7]) in the case of realization of definite conditions. There is an integral transformation of the unknown functions to a system of Riemann invariants which are preserved along the characteristics of the continuous characteristic spectrum. The system admits an infinite number of conservation laws with densities depending on the moments of the distribution function. In addition, the equations of motion can be written in the Hamiltonian form. This makes it possible to find an infinite number of generalized symmetries of the system specified by integrodifferential equations of special form. Here, the exact solutions invariant under the first generalized symmetries from an infinite series have been obtained. Equations that determine the running waves have been integrated completely. For one class of simple waves, the problem of construction of the solution is reduced to a linear integral equation of the second kind resolved by the iteration method.

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REFERENCES

1. G. Russo and P. Smereka , "Kinetic theory for bubble flow. I: Collisionless case," *SIAM J., Appl. Math.*, **56**, 327–357 (1996).
2. L. Van Wijngaarden and C. Kapteyn, "Concentration waves in dilute bubble/liquid mixtures," *J. Fluid Mech.*, **212**, 111–137 (1990).
3. J. B. Kok, "The Fokker–Plank equation for bubble flows and the motion of gas bubble pairs," *Appl. Sci. Res.*, **58**, Nos. 1/4, 319–335 (1997–1998).
4. A. Biesheuvel and W. C. M. Gorissen, "Void fraction disturbance in a uniform bubble liquid," *J. Multuphase Flow*, **16**, 217–231 (1990).
5. L. M. Milne-Thomson, *Theoretical Hydrodynamics*, Macmillan, London (1960).
6. V. M. Teshukov, "Hyperbolicity of long-wave equations," *Dokl. Akad. Nauk SSSR*, **284**, No. 3, 555–559 (1985).
7. V. M. Teshukov, "Long waves in an eddying barotropic fluids," *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 6, 17–26 (1994).
8. F. D. Gakhov, *Boundary-Value Problems* [in Russian], Nauka, Moscow (1977).
9. L. V. Ovsyannikov, *Group Analysis of Differential Equations* [in Russian], Nauka, Moscow (1978).
10. N. Kh. Ibragimov, *Transformation Groups in Mathematical Physics* [in Russian], Nauka, Moscow (1985).
11. P. Olver, *Application of Lie Groups to Differential Equations*, Springer-Verlag, New York–Berlin–Heidelberg–Tokyo (1986).